High-energy asymptotic expansion of the Green function for one-dimensional Fokker–Planck and Schrödinger equations

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Abstract

A new formalism is presented for high-energy analysis of the Green function for Fokker–Planck and Schrödinger equations in one dimension. Formulae for the asymptotic expansion in powers of the inverse wave number are derived, and conditions for the validity of the expansion are studied through the analysis of the remainder term. This method is applicable to a large class of potentials, including the cases where the potential $V(x)$ is infinite as $x \to \pm\infty$. The short-time expansion of the Green function is also discussed.

PACS numbers: 03.65.Nk, 02.30.Hq, 02.50.Ey

1. Introduction

The one-dimensional diffusion in an external potential $V(x)$ is described by the Fokker–Planck equation [1]

$$\frac{\partial}{\partial t} P(x, t) = \frac{\partial^2}{\partial x^2} P(x, t) - 2 \frac{\partial}{\partial x} [f(x) P(x, t)], \tag{1.1}$$

where $f(x)$ is related to the potential $V(x)$ by

$$f(x) = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} V(x). \tag{1.2}$$

With $P(x, t) \equiv e^{-k^2 t} \phi(x)$, equation (1.1) reduces to the time-independent form

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \phi(x) + 2 \frac{\mathrm{d}}{\mathrm{d}x} [f(x) \phi(x)] = k^2 \phi(x). \tag{1.3}$$

In addition to being a fundamental equation for nonequilibrium phenomena, the Fokker–Planck equation is of particular importance in its relation to the Schrödinger equation. Setting

$\phi(x) \equiv e^{-V(x)/2}\psi(x)$, we can transform (1.3) into a steady-state Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V_S(x)\psi(x) = E\psi(x), \tag{1.4}$$

where $E = k^2$, and

$$V_S(x) = f^2(x) + f'(x). \tag{1.5}$$

Historically, methods developed for the Schrödinger equation have been applied to the study of the Fokker–Planck equation. But this should be the other way around as well. Theoretically, the Fokker–Planck equation is often more convenient to deal with than the Schrödinger equation. With the Fokker–Planck equation we can carry out a more systematic analysis, and the results obtained for the Fokker–Planck equation can be applied to quantum-mechanical problems described by the Schrödinger equation. In this paper, we study the high-energy (large-$|k|$) behaviour of the Green function for (1.3) or, equivalently, (1.4). Although we shall mainly work with the Fokker–Planck equation, the results of this paper are directly applicable to the Schrödinger equation, too.

One-dimensional quantum scattering has a long history. In particular, high-energy asymptotic behaviour of the Green function and related functions have been studied over the decades by both physicists and mathematicians. This classical topic has recently attracted attention in connection with inverse problems and the theory of integrable systems, and this area of research remains active even to the present day [2]–[17].

An essential matter in the study of an asymptotic expansion is the estimation of the remainder term. In conventional methods, which are mostly based on an integral equation, it is necessary to impose strong conditions on the potential in order to control the remainder term. For example, the integrability of V_S is often required. For non-integrable V_S, different specific methods need to be used. There has not yet been a formalism in which the asymptotic analysis of the Green function can be carried out systematically and in a unified way for a variety of potentials that may not necessarily be integrable and that may not even be finite as $|x| \to \infty$. In this paper, we take a totally new approach to this problem, and present a new method for the high-energy analysis of the Green function which is applicable to a larger class of potentials.

Our method is based on the analysis of reflection coefficients. Reflection coefficients are fundamental quantities in scattering theory, and they serve as building blocks for constructing the Green function. The Green function can be expressed solely in terms of reflection coefficients for semi-infinite intervals [18]. High-energy behaviour of the reflection coefficients was studied in a previous paper [19], and formulae for their asymptotic expansion were derived there. In the present paper, we apply these results to the Green function, and derive the expansion in powers of the inverse wave number. The coefficients of the expansion are expressed in a simple form in terms of a linear operator, and the remainder term is expressed in terms of transmission and reflection coefficients for finite intervals. The validity of the asymptotic expansion can be studied by using this expression for the remainder term. The short-time expansion of the Green function can also be obtained by this method.

We assume that the (Fokker–Planck) potential $V(x)$ is a real function which is finite for finite x, and which either converges to a finite limit or diverges to infinity ($+\infty$ or $-\infty$) as $x \to \pm\infty$. The quantity k is taken to be a complex number with $\operatorname{Im} k \geqslant 0$.

In sections 2 and 3, we review the relevant results of previous papers. We derive the asymptotic expansion of the Green function in sections 4–6, and discuss the conditions for its validity in sections 7 and 8. In section 9, the short-time expansion is studied. In section 10, we explain how our method can be applied to the Schrödinger equation. Various examples are given in section 11.

2. Reflection coefficients and the Green function

We define the time-dependent Green function $G_F(x, y; t)$ as the solution of

$$\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x} f(x)\right] G_F(x, y; t) = \delta(x - y)\delta(t) \tag{2.1}$$

with the condition that $G_F(x, y; t) = 0$ for $t < 0$. Physically, $G_F(x, y; t)$ is the probability density of finding the diffusing particle at position x at time t, under the condition that it was initially at position y. We also define its Fourier transform

$$G_F(x, y; \omega) \equiv \int_0^\infty e^{i\omega t} G_F(x, y; t) \, dt. \tag{2.2}$$

This $G_F(x, y; \omega)$ is the Green function for equation (1.3) with $k^2 = i\omega$. In the same way, we define the retarded Green function for the Schrödinger equation as the solution of

$$\left[i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - V_S(x)\right] G_S(x, y; t) = \delta(x - y)\delta(t) \tag{2.3}$$

satisfying the condition $G_S(x, y; t) = 0$ for $t < 0$. Its Fourier transform defined by

$$G_S(x, y; E) \equiv \int_0^\infty e^{iEt} G_S(x, y; t) \, dt \tag{2.4}$$

is the Green function for the steady-state Schrödinger equation (1.4), satisfying

$$\left[\frac{\partial^2}{\partial x^2} - V_S(x) + E\right] G_S(x, y; E) = \delta(x - y). \tag{2.5}$$

It is easy to see that G_F and G_S are related by

$$G_F(x, y; \omega) = -e^{-[V(x)-V(y)]/2} G_S(x, y; E = i\omega). \tag{2.6}$$

For convenience, we define

$$G(x, y; k) \equiv 2ik G_S(x, y; E = k^2) \tag{2.7}$$

as a function of complex k, and deal with this G instead of G_F or G_S. (We shall refer to G as the Green function, too.) Since $G(x, y; k) = G(y, x; k)$, without loss of generality we assume that $x \geqslant y$.

To introduce our basic expression for the Green function, let us first define the transmission and reflection coefficients for finite intervals. We consider an interval (x_1, x_2), and define

$$\bar{V}(x) \equiv \begin{cases} V(x_1) & (x < x_1) \\ V(x) & (x_1 \leqslant x \leqslant x_2), \\ V(x_2) & (x_2 < x) \end{cases} \qquad \bar{f}(x) \equiv -\frac{1}{2}\frac{d}{dx}\bar{V}(x). \tag{2.8}$$

Namely, $\bar{V}(x)$ is identical to $V(x)$ within (x_1, x_2) and constant outside this interval. We consider equation (1.3) with $f(x)$ replaced by $\bar{f}(x)$:

$$-\frac{d^2}{dx^2}\phi(x) + 2\frac{d}{dx}[\bar{f}(x)\phi(x)] = k^2\phi(x). \tag{2.9}$$

(In general, the left-hand side of this equation contains delta functions at $x = x_1$ and $x = x_2$ coming from the derivative of $\bar{f}$.) Since $\bar{f}(x) = 0$ for $x < x_1$ and $x > x_2$, equation (2.9) has two solutions of the form

$$\phi_1(x) = \begin{cases} \tau(x_2, x_1; k) \, e^{-ik(x-x_1)} & x < x_1, \\ e^{-ik(x-x_2)} + R_r(x_2, x_1; k) \, e^{ik(x-x_2)} & x > x_2, \end{cases} \tag{2.10a}$$

$$\phi_2(x) = \begin{cases} e^{ik(x-x_1)} + R_l(x_2, x_1; k)\, e^{-ik(x-x_1)} & x < x_1, \\ \tau(x_2, x_1; k)\, e^{ik(x-x_2)} & x > x_2. \end{cases} \tag{2.10b}$$

The transmission coefficient τ, the right reflection coefficient R_r, and the left reflection coefficient R_l for the interval (x_1, x_2) are defied by (2.10).

We can let $x_1 \to -\infty$ in (2.10a) or $x_2 \to +\infty$ in (2.10b) to define the reflection coefficients for semi-infinite intervals:

$$R_r(x_2, -\infty; k) = \lim_{x_1 \to -\infty} R_r(x_2, x_1; k), \qquad R_l(\infty, x_1; k) = \lim_{x_2 \to +\infty} R_l(x_2, x_1; k). \tag{2.11}$$

Let us define

$$S_r(x, k) \equiv \frac{R_r(x, -\infty; k)}{1 + R_r(x, -\infty; k)}, \qquad S_l(x, k) \equiv \frac{R_l(\infty; x; k)}{1 + R_l(\infty, x; k)}, \tag{2.12}$$

and

$$S(x, k) \equiv S_r(x, k) + S_l(x, k). \tag{2.13}$$

It was shown in [18] that the Green function can be expressed in terms of this S as

$$G(x, y; k) = \frac{1}{\sqrt{[1 - S(x, k)][1 - S(y, k)]}} \exp\left[ik(x - y) - ik \int_y^x S(z, k)\,dz\right]. \tag{2.14}$$

The logarithm of (2.14) reads

$$\log G(x, y; k) = ik(x - y) - ik \int_y^x S(z, k)\,dz - \frac{1}{2}\log[1 - S(x, k)] - \frac{1}{2}\log[1 - S(y, k)]. \tag{2.15}$$

We shall carry out the analysis of the Green function on the basis of this expression.

3. High-energy expansion formula for the reflection coefficients

In this section, we review the formulae derived in [19] for the high-energy expansion of the reflection coefficients. Here we deal only with R_r. (Corresponding formulae for R_l can be obtained in the same way.) First, we define the generalized transmission and reflection coefficients, with an additional argument ξ, as

$$\bar{R}_r(x, y; \xi; k) \equiv \frac{R_r(x, y; k) - \xi}{1 - \xi R_r(x, y; k)}, \tag{3.1a}$$

$$\bar{R}_l(x, y; \xi; k) \equiv R_l(x, y; \xi; k) + \frac{\xi \tau^2(x, y; \xi; k)}{1 - \xi R_r(x, y, \xi; k)}, \tag{3.1b}$$

$$\bar{\tau}(x, y; \xi; k) \equiv \frac{\sqrt{1 - \xi^2}\,\tau(x, y; k)}{1 - \xi R_r(x, y; k)}. \tag{3.1c}$$

We also define the operator $\mathcal{M}$ which acts on functions of x and ξ as

$$\mathcal{M}g(x, \xi) \equiv -f(x)[(\xi - \xi^{-1})g(x, \xi) + \xi^{-1}g(x, 0)] + \xi^{-1}\int_0^\xi d\xi\, \frac{\partial}{\partial x} g(x, \xi), \tag{3.2}$$

where $g(x, \xi)$ is an arbitrary function, and $f(x)$ is the function in (1.1)–(1.3).

It was shown in [19] that, for an arbitrary nonnegative integer N,

$$\bar{R}_r(x, -\infty; \xi; k) = -\xi + \frac{1}{2ik}\bar{c}_1(x, \xi) + \frac{1}{(2ik)^2}\bar{c}_2(x, \xi) + \cdots$$

$$+ \frac{1}{(2ik)^N}\bar{c}_N(x, \xi) + \bar{\delta}_N(x, \xi, k), \tag{3.3}$$

where

$$\bar{c}_n(x, \xi) = -(1 - \xi^2)\mathcal{M}^{n-1} f(x), \tag{3.4}$$

$$\bar{\delta}_N(x, \xi; k) = \frac{1}{(2\mathrm{i}k)^N} \int_{-\infty}^{x} \bar{\tau}^2(x, z; \xi; k) K_N(z, \bar{R}_l(x, z; \xi; k)) \, \mathrm{d}z, \tag{3.5}$$

$$K_n(x, \xi) \equiv -\left(1 + \xi \frac{\partial}{\partial \xi}\right) \frac{1}{1 - \xi^2} \bar{c}_{n+1}(x, \xi). \tag{3.6}$$

In (3.5), $K_N(z, \bar{R}_l(x, z; \xi; k))$ is the quantity obtained from $K_N(z, \xi)$ by the substitution $\xi \to \bar{R}_l(x, z; \xi; k)$. It is convenient to define

$$\tilde{c}_n(x, \xi) \equiv \frac{1}{1 - \xi^2} \bar{c}_n(x, \xi), \tag{3.7}$$

so that (3.4) and (3.6) read

$$\tilde{c}_n(x, \xi) = -\mathcal{M}^{n-1} f(x), \tag{3.8}$$

$$K_n(x, \xi) = -\left(1 + \xi \frac{\partial}{\partial \xi}\right) \tilde{c}_{n+1}(x, \xi). \tag{3.9}$$

It is easy to calculate $\tilde{c}_n$ from (3.8) by using definition (3.2). We have

$$\tilde{c}_1 = -f, \qquad \tilde{c}_2 = -f' + f^2 \xi, \qquad \tilde{c}_3 = -f'' + f^3 + 2ff'\xi - f^3\xi^2,$$
$$\tilde{c}_4 = -f''' + 5f^2 f' - (2f^4 - f'^2 - 2ff'')\xi - 3f^2 f'\xi^2 + f^4\xi^3, \text{ etc.} \tag{3.10}$$

Obviously $\tilde{c}_n$ is an $(n-1)$th-degree polynomial in ξ, whose coefficients consist of the powers of f and its derivatives. The K_n's are obtained from (3.9) and (3.10) as

$$K_0 = f, \qquad K_1 = f' - 2f^2\xi, \qquad K_2 = f'' - f^3 - 4ff'\xi + 3f^3\xi^2,$$
$$K_3 = f''' - 5f^2 f' + 2(2f^4 - f'^2 - 2ff'')\xi + 9f^2 f'\xi^2 - 4f^4\xi^3, \text{ etc.} \tag{3.11}$$

Equations (3.3)–(3.6) hold for any $f(x)$ as long as $\bar{c}_n$ and $\bar{\delta}_N$ make sense. However, (3.3) is meaningful as a high-energy expression only if

$$\lim_{|k| \to \infty} k^N \bar{\delta}_N(x, \xi, k) = 0. \tag{3.12}$$

Using (3.5) with (3.9), we can study the conditions for (3.12) to hold. For simplicity, let us assume that $f(x)$ and all its derivatives are monotone for sufficiently large $|x|$. (This condition is unnecessarily strong and can be relaxed, but we make this assumption in order to simplify the presentation.) Then it can be shown[1] that

(i) When $|k| \to \infty$ with fixed $\arg k$ in the range $0 < \arg k < \pi$, equation (3.12) holds if (f is $(N-1)$ times differentiable and) $f^{(N-1)}$ is continuous and piecewise smooth[2].

(ii) When $|k| \to \infty$ with fixed $\mathrm{Im}\, k > 0$, equation (3.12) holds if $f^{(N-1)}$ is continuous and piecewise smooth, and if $\lim_{z \to -\infty} f(z)e^{cz} = 0$ for any positive number c.

(iii) When $|k| \to \infty$ with $\mathrm{Im}\, k = 0$, equation (3.12) holds if $f^{(N-1)}$ is continuous and piecewise smooth, and if $f(-\infty)$ is finite.

This means that if the conditions in (i), (ii) or (iii) hold, then (3.12) holds in the region (i) $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$, (ii) $\mathrm{Im}\, k \geqslant \epsilon$, (iii) $\mathrm{Im}\, k \geqslant 0$, respectively, where ϵ is an arbitrary positive number. (We shall always let ϵ stand for a positive quantity.)

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[1] The proof is given in [19] only for $\xi = 0$, but this is sufficient. (See the comment below equation (5.16) of [19].) If (3.12) holds for $\xi = 0$ then it holds for any ξ. This can be easily shown by substituting the expansion of R_r (equation (3.3) with $\xi = 0$) into the right-hand side of (3.1a).

[2] This last condition is described as 'piecewise differentiable' in [19], but it should properly be 'piecewise continuously differentiable' or 'piecewise smooth' to avoid some pathological situations.

4. Expansion of S

We wish to derive the $1/k$-expansion of $\log G$ from (2.15) by using the formulae given in the previous section. To do so, we need to study the expansion of S and $\log(1 - S)$. From (2.12) and (3.1a), we find

$$S_r(x, k) = \lim_{\xi \to -1} \frac{1}{1 - \xi^2} [\bar{R}_r(x, -\infty; \xi; k) + \xi]. \tag{4.1}$$

Therefore, the expansion of S_r is obtained by substituting (3.3) into (4.1) as

$$S_r(x, k) = \frac{1}{2ik} s_1(x) + \frac{1}{(2ik)^2} s_2(x) + \cdots + \frac{1}{(2ik)^N} s_N(x) + \sigma_N^r(x, k), \tag{4.2}$$

where, using definition (3.7),

$$s_n(x) = \tilde{c}_n(x, -1), \tag{4.3}$$

$$\sigma_N^r(x, k) = \lim_{\xi \to -1} \frac{1}{1 - \xi^2} \bar{\delta}_N(x, \xi, k). \tag{4.4}$$

Setting $\xi = -1$ in (3.10), we have

$$s_1 = -f, \qquad s_2 = -f' - f^2, \qquad s_3 = -f'' - 2ff',$$
$$s_4 = -f''' - 2ff'' - f'^2 + 2f^2 f' + f^4$$
$$s_5 = -f^{(4)} - 2ff''' - 2f'f'' + 4f^2 f'' + 8f(f')^2 + 8f^3 f', \text{ etc.} \tag{4.5}$$

Substituting (3.5) with (3.1) into (4.4) gives the expression for σ_N^r,

$$\sigma_N^r(x, k) = \frac{1}{(2ik)^N} \int_{-\infty}^x \left[\frac{\tau(x, z; k)}{1 + R_r(x, z; k)} \right]^2 K_N(z, \eta_l) \, dz, \tag{4.6}$$

where we have defined

$$\eta_l \equiv R_l(x, z; k) - \frac{\tau^2(x, z; k)}{1 + R_r(x, z; k)}. \tag{4.7}$$

The expressions for S_l corresponding to (4.2), (4.6) and (4.7) can be derived in the same way, using the analogues of the formulae of section 3 for R_l. The result is

$$S_l(x, k) = \frac{1}{(-2ik)} s_1(x) + \frac{1}{(-2ik)^2} s_2(x) + \cdots + \frac{1}{(-2ik)^N} s_N(x) + \sigma_N^l(x, k), \tag{4.8}$$

$$\sigma_N^l(x, k) = \frac{-1}{(-2ik)^N} \int_x^\infty \left[\frac{\tau(z, x; k)}{1 + R_l(z, x; k)} \right]^2 K_N(z, \eta_r) \, dz, \tag{4.9}$$

$$\eta_r \equiv R_r(z, x; k) - \frac{\tau^2(z, x; k)}{1 + R_l(z, x; k)}. \tag{4.10}$$

The coefficients s_n in (4.8) are the same ones as in (4.2). The expansion of $S(x, k)$ (equation (2.13)) is obtained by adding (4.2) and (4.8):

$$S(x, k) = 2 \left[\frac{1}{(2ik)^2} s_2(x) + \frac{1}{(2ik)^4} s_4(x) + \cdots + \frac{1}{(2ik)^{N'}} s_{N'}(x) \right] + \sigma_N(x, k), \tag{4.11}$$

$$\sigma_N(x, k) \equiv \sigma_N^r(x, k) + \sigma_N^l(x, k), \tag{4.12}$$

where

$$N' \equiv \begin{cases} N & (N \text{ even}) \\ N - 1 & (N \text{ odd}). \end{cases} \tag{4.13}$$

Note that $\sigma_{N+1} = \sigma_N$ for any even number N.

Obviously, a sufficient condition for

$$\lim_{|k|\to\infty} k^N \sigma_N(x, k) = 0 \tag{4.14}$$

is that the following two equations hold:

$$\lim_{|k|\to\infty} k^N \sigma_N^r(x, k) = 0, \qquad \lim_{|k|\to\infty} k^N \sigma_N^l(x, k) = 0. \tag{4.15}$$

From (4.4) we can see that the first equation of (4.15) holds if (3.12) holds. (The limit $\xi \to -1$ does not interfere with the limit $|k| \to \infty$. We can also show this directly by substituting the expansion of R_r into the first equation of (2.12).) Therefore, the first equation of (4.15) holds under the conditions given in section 3. The conditions for the second equation are obvious analogues. Hence it is apparent that (4.14) holds in the sector $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$ if $f^{(N-1)}$ is continuous and piecewise continuous differentiable. If, in addition, both $f(+\infty)$ and $f(-\infty)$ are finite, then (4.14) holds for $0 \leqslant \arg k \leqslant \pi$.

5. Expansion of $\log(1 - S)$

From (4.11) we can derive the expansion of $\log(1 - S)$ as

$$-\frac{1}{2} \log[1 - S(x, k)] = \frac{1}{(2ik)^2} \alpha_2(x) + \frac{1}{(2ik)^4} \alpha_4(x) + \cdots + \frac{1}{(2ik)^{N'}} \alpha_{N'}(x) + \rho_N(x, k), \tag{5.1}$$

where the coefficients $\alpha_2, \alpha_4, \alpha_6, \ldots$ are expressed in terms of $\{s_n\}$ as

$$\alpha_{2i} = \frac{1}{2} \sum_{m=1}^{\infty} \frac{2^m}{m} \sum_{\substack{\{j_1,\ldots,j_m\} \\ \Sigma j_n = i}} s_{2j_1} s_{2j_2} \cdots s_{2j_m}. \tag{5.2}$$

(The second sum in (5.2) is over $j_n = 1, 2, 3, \ldots$ for each n ($1 \leqslant n \leqslant m$) with the constraint $\sum_{n=1}^{m} j_n = i$.) The remainder term ρ_N of (5.1) can be written as

$$\rho_N = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\sigma_N}{1 - A_N} \right)^m + \left[-\frac{1}{2} \log(1 - A_N) - \sum_{n=1}^{N'/2} \frac{\alpha_{2n}}{(2ik)^{2n}} \right], \tag{5.3}$$

where $A_N \equiv 2 \sum_{n=1}^{N'/2} s_{2n}/(2ik)^{2n}$, so that $S = A_N + \sigma_N$ (see (4.11)). The quantity in the brackets on the right-hand side of (5.3) is $O(1/|k|^{N+1})$ as $|k| \to \infty$. Hence

$$\rho_N = \frac{1}{2} \sigma_N[1 + o(1)] + O(1/|k|^{N+1}) \qquad (|k| \to \infty) \tag{5.4}$$

provided that σ_N vanishes as $|k| \to \infty$. Note also that

$$\rho_N = \rho_{N+1} \quad (N \text{ even}), \qquad \rho_N = \rho_{N+1} + \frac{\alpha_{N+1}}{(2ik)^{N+1}} \quad (N \text{ odd}). \tag{5.5}$$

The coefficients $\alpha_2, \alpha_4, \ldots$ of (5.1), which are given by (5.2), can be expressed in a more compact form. From equations (A.3) of appendix A, we obtain

$$-\frac{1}{2} \frac{\partial}{\partial x} \log[1 - S(x, k)] = f(x) + ik[S_r(x, k) - S_l(x, k)]. \tag{5.6}$$

Substituting (4.2) and (4.8) into the right-hand side of (5.6) yields

$$-\frac{1}{2} \frac{\partial}{\partial x} \log[1 - S(x, k)] = \frac{1}{(2ik)^2} s_3(x) + \frac{1}{(2ik)^4} s_5(x) + \frac{1}{(2ik)^6} s_7(x) + \cdots, \tag{5.7}$$

where we have used $s_1 = -f$. Comparing (5.1) with (5.7) we find that $(\mathrm{d}/\mathrm{d}x)\alpha_n(x) = s_{n+1}(x)$ for any even number n. Integrating both sides of this equation gives

$$\alpha_n(x) = \int^x s_{n+1}(z)\,\mathrm{d}z. \tag{5.8}$$

The integral on the right-hand side is uniquely determined in such a way that α_n includes no additional integration constant when expressed in terms of f and its derivatives. For example, $s_3 = -(f^2 + f')'$, $s_5 = (2f^4 + 4f^2 f' - 2ff'' + f^{(3)})'$, as can be seen from (4.5). In this way, s_m is a total derivative for odd $m \geqslant 3$. Hence we have

$$\alpha_2 = -f^2 - f', \qquad \alpha_4 = 2f^4 + 4f^2 f' - 2ff'' + f^{(3)}, \text{ etc.} \tag{5.9}$$

If $f(-\infty) = 0$ or $f(+\infty) = 0$, we can write the right-hand side of (5.8) as $\int_{-\infty}^x s_{n+1}(z)\,\mathrm{d}z$ or $-\int_x^\infty s_{n+1}(z)\,\mathrm{d}z$. But (5.8) holds even if neither $f(-\infty) = 0$ nor $f(+\infty) = 0$. Equations (5.1) and (5.6) also give the relation between the remainder terms

$$\frac{\partial}{\partial x}\rho_N(x, k) = \mathrm{i}k\left[\sigma_{N+1}^r(x, k) - \sigma_{N+1}^l(x, k)\right]. \tag{5.10}$$

6. The $1/k$-expansion of $\log G$

Substituting (4.11), (5.1) and (5.8) into (2.15), we obtain, for any integer $N \geqslant 0$,

$$\log G(x, y; k) = \mathrm{i}k(x - y) + \frac{1}{2\mathrm{i}k}a_1(x, y) + \frac{1}{(2\mathrm{i}k)^2}a_2(x, y) + \frac{1}{(2\mathrm{i}k)^3}a_3(x, y)$$

$$+ \cdots + \frac{1}{(2\mathrm{i}k)^N}a_N(x, y) + \Delta_N(x, y; k), \tag{6.1}$$

where

$$a_n(x, y) = -\int_y^x s_{n+1}(z)\,\mathrm{d}z \quad (n \text{ odd}), \tag{6.2a}$$

$$a_n(x, y) = \int^x s_{n+1}(z)\,\mathrm{d}z + \int^y s_{n+1}(z)\,\mathrm{d}z \quad (n \text{ even}), \tag{6.2b}$$

$$\Delta_N(x, y; k) = -\mathrm{i}k\int_y^x \sigma_{N+1}(z, k)\,\mathrm{d}z + \rho_N(x, k) + \rho_N(y, k). \tag{6.3}$$

We can also write the right-hand side of (6.2b) as $\alpha_n(x) + \alpha_n(y)$ with α_n given by (5.2).

For (6.1) to be meaningful as a high-energy expansion, Δ_N must satisfy

$$\lim_{|k|\to\infty} k^N \Delta_N(x, y; k) = 0. \tag{6.4}$$

Let us study the conditions for (6.4) using expression (6.3). We first remark that (5.4) and (5.5) imply

$$\rho_N = \tfrac{1}{2}\sigma_{N+1}[1 + o(1)] + O(1/|k|^{N+1}) \qquad (|k| \to \infty). \tag{6.5}$$

Now suppose that $\lim_{|k|\to\infty} k^{N+1}\sigma_{N+1} = 0$. Then,

$$\lim_{|k|\to\infty} k^{N+1}\int_y^x \sigma_{N+1}(z, k)\,\mathrm{d}z = 0, \qquad \lim_{|k|\to\infty} k^N \rho_N = 0, \tag{6.6}$$

and so (6.4) holds. (Since $|\sigma_{N+1}(z, k)|$ is uniformly bounded in the interval $y < z < x$, the limit $|k| \to \infty$ and the integral can be interchanged. The second equation follows from (6.5).) We know that $\lim_{|k|\to\infty} k^{N+1}\sigma_{N+1} = 0$ is satisfied under the conditions stated at the end of section 4, with N replaced by $N + 1$. Hence it follows that (6.4) holds for $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$ if $f^{(N)}$ is continuous and piecewise continuously differentiable. It holds for $0 \leqslant \arg k \leqslant \pi$ if, in addition, both $f(-\infty)$ and $f(+\infty)$ are finite.

7. Validity of (6.4) for discontinuous $f^{(N)}$

The conditions for (6.4) mentioned at the end of the last section are sufficient conditions, not necessary ones. Here we show that $f^{(N)}$ need not be continuous for (6.4) to hold.

Suppose that $f(z)$ is $(M-1)$ times continuously differentiable, $f^{(M-1)}(z)$ is continuously differentiable except at $z = 0$, and that $f^{(M)}$ has a finite jump at $z = 0$:

$$f^{(M)}(z) = C\theta(z) + \cdots. \tag{7.1}$$

Here $\theta(z)$ is the Heaviside step function, and C is a constant. Then $f^{(M+1)}$ contains a delta function as $f^{(M+1)}(z) = C\delta(z) + \cdots$. It is easy to show that K_n (equation (3.9)) has the form $K_n = f^{(n)} + \cdots$, where the remaining terms on the right-hand side do not contain derivatives of order n or higher. (See (3.11).) Therefore,

$$K_{M+1}(z, \xi) = C\delta(z) + \cdots. \tag{7.2}$$

Substituting this into (4.6) and (4.9) with $N = M + 1$, and then into (4.12), we have

$$\begin{aligned}
\sigma_{M+1}(z, k) &= \frac{C}{(2ik)^{M+1}} \int_{-\infty}^{z} \left[\frac{\tau(z, w)}{1 + R_r(z, w)}\right]^2 \delta(w)\,dw \\
&\quad - \frac{C}{(-2ik)^{M+1}} \int_{z}^{\infty} \left[\frac{\tau(w, z)}{1 + R_l(w, z)}\right]^2 \delta(w)\,dw + \cdots \\
&= \frac{C}{(2ik)^{M+1}} \left\{\left[\frac{\tau(z, 0)}{1 + R_r(z, 0)}\right]^2 \theta(z) + (-1)^M \left[\frac{\tau(0, z)}{1 + R_l(0, z)}\right]^2 \theta(-z)\right\} + \cdots. \tag{7.3}
\end{aligned}$$

The quantity in the curly brackets in the last line of (7.3) does not vanish as $|k| \to \infty$ when k is real. (See (A.2).) But its integral does vanish in this limit even when $\mathrm{Im}\,k = 0$. Indeed, using (A.5), (A.1) and (A.2) of appendix A we find

$$\begin{aligned}
\int_{y}^{x} &\left\{\left[\frac{\tau(z, 0)}{1 + R_r(z, 0)}\right]^2 \theta(z) + (-1)^M \left[\frac{\tau(0, z)}{1 + R_l(0, z)}\right]^2 \theta(-z)\right\} dz \\
&= \frac{1}{2ik}[(e^{2ikx} - e^{2ik\max(y,0)})\theta(x) + (-1)^M (e^{-2iky} - e^{-2ik\min(x,0)})\theta(-y)] \\
&\quad + o(1/|k|) \qquad (|k| \to \infty, 0 \leqslant \arg k \leqslant \pi). \tag{7.4}
\end{aligned}$$

Assuming that the part represented by '$\cdots$' in (7.3) is $o(1/|k|^{M+1})$, we have

$$\sigma_{M+1}(z, k) = O(1/|k|^{M+1}), \qquad \int_{y}^{x} \sigma_{M+1}(z, k)\,dz = o(1/|k|^{M+1}), \tag{7.5}$$

and, from (6.5),

$$\rho_M(z, k) = O(1/|k|^{M+1}). \tag{7.6}$$

From (6.3), (7.5) and (7.6) it follows that $\Delta_M = o(1/|k|^M)$, so (6.4) holds for $N = M$ in spite of the discontinuity of $f^{(M)}$.

The same argument holds when $f^{(M)}$ has two or more finite jumps. In summary, the following conclusion can be drawn.

(i) If f is $(N-1)$ times continuously differentiable and $f^{(N-1)}$ is piecewise continuously differentiable, and if $f^{(N)}$ is piecewise continuously differentiable, then (6.4) holds in the angular region $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$ with arbitrary $\epsilon > 0$. (Here $f^{(N)}$ may possibly have a finite number of finite jumps.)

(ii) If the conditions of (i) are satisfied, and if $\lim_{z\to+\infty} f(z)e^{-cz} = \lim_{z\to-\infty} f(z)e^{cz} = 0$ for any $c > 0$, then (6.4) holds in the half plane $\mathrm{Im}\,k \geqslant \epsilon$ with arbitrary $\epsilon > 0$.

(iii) If the conditions of (i) are satisfied, and if both $f(+\infty)$ and $f(-\infty)$ are finite, then (6.4) holds in the upper half plane $\mathrm{Im}\, k \geqslant 0$ including the real axis.

(As noted below equation (3.12), we are assuming that $f(z)$ and its derivatives are monotone for sufficiently large $|z|$, at both $z \to +\infty$ and $z \to -\infty$.)

The conditions of (i) are still not necessary conditions. Actually, (6.4) may hold even when f is not $(N-1)$ times differentiable, as will be shown in the following section.

8. Effects of the jump of $f^{(M)}$ at higher orders

Here we show that (6.4) may hold even for $N \geqslant M+1$ when $f^{(M)}$ has a discontinuity as in (7.1). To this end, let us study $\Delta_{M+1}, \Delta_{M+2}, \ldots$ for the case of (7.1). Since $K_{M+2} = f^{(M+2)} + \cdots$, we have

$$K_{M+2}(z, \xi) = C\delta'(z) + \cdots, \tag{8.1}$$

where δ' is the derivative of the delta function. As before, we substitute (8.1) into (4.6) and (4.9), and then into (4.12). After integrating by parts, we obtain

$$\sigma_{M+2}(z, k) = \frac{C}{(2\mathrm{i}k)^{M+2}} \left\{ \delta(z) - \frac{\partial}{\partial w}\left[\frac{\tau(z, w)}{1 + R_r(z, w)} \right]^2 \Bigg|_{w=0} \theta(z) \right\}$$

$$+ \frac{C}{(-2\mathrm{i}k)^{M+2}} \left\{ \delta(z) + \frac{\partial}{\partial w}\left[\frac{\tau(w, z)}{1 + R_l(w, z)} \right]^2 \Bigg|_{w=0} \theta(-z) \right\} + \cdots. \tag{8.2}$$

Using (A.4) of appendix A, we can write (8.2) as

$$\sigma_{M+2}(z, k) = \frac{C}{(2\mathrm{i}k)^{M+1}} \left\{ \left[\frac{\tau(z, 0)}{1 + R_r(z, 0)} \right]^2 \theta(z) + (-1)^M \left[\frac{\tau(0, z)}{1 + R_l(0, z)} \right]^2 \theta(-z) \right\}$$

$$+ \frac{2C}{(2\mathrm{i}k)^{M+2}} A_M \delta(z) + \cdots, \tag{8.3}$$

where

$$A_M \equiv \begin{cases} 1 & (M \text{ even}) \\ 0 & (M \text{ odd}). \end{cases} \tag{8.4}$$

In (8.3), we have disregarded the terms of order $1/k^{M+2}$ which have the form $Q(z, k)/(2\mathrm{i}k)^{M+2}$ with $Q(z, k)$ such that $\int_y^x Q(z, k)\,\mathrm{d}z$ vanishes as $|k| \to \infty$. Such terms are irrelevant to the following discussion. In addition to $C\delta'(z)$, the right-hand side of (8.1) contains another singular term proportional to $f(z)\delta(z)$, but the contribution from this term can be disregarded for the same reason. We assume that the contribution to (8.3) from non-singular terms of K_{M+2} is $o(1/|k|^{M+2})$ as $|k| \to \infty$.

Now we consider the two cases, $y < 0 < x$ and $0 < y < x$.

(a) $y < 0 < x$.
The integral of (8.3) can be calculated by using (7.4). For the case $y < 0 < x$, we have

$$\int_y^x \sigma_{M+2}(z, k)\,\mathrm{d}z = \frac{C}{(2\mathrm{i}k)^{M+2}}[\mathrm{e}^{2\mathrm{i}kx} + (-1)^M \mathrm{e}^{-2\mathrm{i}ky}] + o(1/|k|^{M+2}). \tag{8.5}$$

(Note that the contribution from the delta function in (8.3) is cancelled). From (5.4), (7.3) and (A.2) (or from (6.5), (8.3) and (A.2)) we obtain

$$\rho_{M+1}(x, k) = \frac{C}{2(2\mathrm{i}k)^{M+1}} \mathrm{e}^{2\mathrm{i}kx} + o(1/|k|^{M+1}), \tag{8.6a}$$

$$\rho_{M+1}(y, k) = (-1)^M \frac{C}{2(2ik)^{M+1}} \, e^{-2iky} + o(1/|k|^{M+1}).$$ (8.6b)

On substituting (8.5), (8.6a) and (8.6b) into (6.3) with $N = M + 1$, the terms of order $1/k^{M+1}$ cancel out, and we have

$$\Delta_{M+1}(x, y; k) = o(1/|k|^{M+1}) \qquad (|k| \to \infty).$$ (8.7)

Namely, (6.4) holds for $N = M + 1$.

The effect of the discontinuity of $f^{(M)}$ on $\Delta_{M+2}, \Delta_{M+3}$ etc can be studied in the same way. Singular terms involving derivatives of the delta function such as $C\delta''$, $C\delta'''$, ... appear in $K_{M+3}, K_{M+4}, \ldots$, respectively, but they cause no problems. Singular contributions coming from these terms cancel out in the expressions for $\Delta_{M+2}, \Delta_{M+3}$ etc, as we have seen above for Δ_{M+1}. (The explanation is omitted here, but the cancellation between $ik \int \sigma_{N+1} \, dz$ and ρ_N is guaranteed by relation (5.10).) As a result, the derivatives of the delta function do not produce a term of order $1/k^N$ (or lower) in Δ_N. In (6.2) and (6.3), the delta function and its derivatives appear only within an integral, and these expressions are well defined[3] as long as $n, N \leqslant 2M + 1$.

However, (6.3) is not well defined for $N = 2M + 2$ since the square of the delta function appears in it. We can easily see that K_{2M+3} contains a term proportional to $(f^{(M+1)})^2 = C^2\delta^2$. In addition, terms proportional to $f^{(M+1+i)} f^{(M+1-i)}$ $(i = 1, 2, \ldots)$ are contained in K_{2M+3}. These terms are ill defined, too, since they produce the square of the delta function by integration by parts. To study Δ_{2M+2}, we need to go back one step, as we cannot directly use (6.3). By induction, it can be shown that

$$K_{2M+2} = f^{(2M+2)} - 4\xi \sum_{i=1}^{M+1} f^{(M+i)} f^{(M+1-i)} + \cdots.$$ (8.8)

The terms explicitly shown in (8.8) are the ones that give rise to the ill-defined terms in K_{2M+3}. We substitute (8.8) into the expression for Δ_{2M+1}. Then we can study Δ_{2M+2} by using this expression together with the relation $\Delta_{2M+2} = \Delta_{2M+1} - a_{2M+2}/(2ik)^{2M+2}$. After some calculation[4] we find that Δ_{2M+2} contains a term of order $1/k^{2M+2}$ as

$$\Delta_{2M+2} = \frac{1}{(2ik)^{2M+2}} \frac{(-1)^M}{2} C^2 + \cdots.$$ (8.9)

So (6.4) does not hold for $N = 2M + 2$. We can see that (8.9) gives correction to the coefficient of order $2M + 2$ as

$$\log G = ik(x - y) + \frac{1}{2ik} a_1 + \cdots + \frac{1}{(2ik)^{2M+2}} [a_{2M+2} + (-1)^M C^2/2] + \cdots.$$ (8.10)

Thus, the correct coefficient of order $2M + 2$ is not a_{2M+2} but $a_{2M+2} + (-1)^M C^2/2$.

In summary, when $f^{(M)}$ has a finite jump of the form (7.1), and when the jump is located between y and x, equation (6.4) holds for $N \leqslant 2M + 1$ in the sector $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$ provided that the part represented by '$\cdots$' in (7.1) is sufficiently differentiable. If, in addition,

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[3] Needless to say, $\int_a^b g(z)\delta^{(m)}(z)dz = (-1)^m g^{(m)}(0)$ for $a < 0 < b$. In (6.2) and (6.3), the function multiplying $\delta^{(m)}$ is always m times continuously differentiable if $n \leqslant 2M + 1$ (for (6.2)) or $N \leqslant 2M$ (for (6.3)). An expression like $\int_a^b \theta(z)g(z)\delta(z)dz = \theta(0)g(0)$ may appear in Δ_{2M+1}, where $\theta(0) = 1/2$.

[4] It turns out that only the first term on the right-hand side of (8.8) is relevant. Anomalous contribution comes from where $\theta(0)$ appears in formal calculation whereas it should really be $\theta(0 + \epsilon)$ or $\theta(0 - \epsilon)$. For example, $ik \int_0^x \theta(z)g(z)e^{ikz}dz = -g(0) + g(x)e^{ikx} + o(1)$ $(|k| \to \infty)$, where the first term on the right-hand side is not $-\theta(0)g(0) = -g(0)/2$ but $-\theta(0 + \epsilon)g(0)$. In deriving (8.9), we also use the $1/k$-expansion of R_r, R_l and τ for finite intervals. (See the comments below (A.2) in appendix A.) In the expansion, we need only to keep track of the terms which are linear in f and its derivatives.

$f(-\infty)$ and $f(+\infty)$ are both finite, then (6.4) holds for $N \leqslant 2M + 1$ even for real k. In any case, (6.4) does not hold for $N \geqslant 2M + 2$. Although $\log G$ can be asymptotically expanded in powers of $1/k$ even beyond the term of order $1/k^{2M+2}$, the coefficients a_n given by (6.2) are not correct for $n \geqslant 2M + 2$. In particular, a_{2M+2} is shifted by $(-1)^M C^2/2$ as in (8.10). (See examples 5–8 of section 11.)

(b) $0 < y < x$.
Similarly, for $0 < y < x$ we have, by using (7.4),

$$\int_y^x \sigma_{M+2}(z, k) \, dz = \frac{C}{(2ik)^{M+2}} (e^{2ikx} - e^{2iky}) + o(1/|k|^{M+2}), \tag{8.11}$$

$$\rho_{M+1}(y, k) = \frac{C}{2(2ik)^{M+1}} e^{2iky} + o(1/|k|^{M+1}), \tag{8.12}$$

while the expression for $\rho_{M+1}(x, k)$ is the same as (8.6a). Substituting (8.11), (8.6a) and (8.12) into (6.3), we obtain

$$\Delta_{M+1}(x, y; k) = \frac{C}{(2ik)^{M+1}} e^{2iky} + o(1/|k|^{M+1}) \qquad (|k| \to \infty). \tag{8.13}$$

If the limit $|k| \to \infty$ is taken with fixed $\arg k$ in the sector $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$, then $\Delta_{M+1} = o(1/|k|^{M+1})$ since e^{2iky} falls off exponentially. The same can be said for Δ_{M+2}, Δ_{M+3}, and so on; the contribution to Δ_N coming from the discontinuity of $f^{(M)}$ is exponentially small at large $|k|$ as long as $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$.

When k is real, however, $k^{M+1}\Delta_{M+1}$ does not vanish as $|k| \to \infty$. So, unlike case (a), equation (6.4) does not hold for $N \geqslant M + 1$ when the limit is taken along the real axis. (See example 6 of section 11.)

The case $y < x < 0$ can be treated in exactly the same way as (b). It is straightforward to extend the arguments of (a) and (b) above to the cases where there are two or more such discontinuities.

9. Short-time expansion of the Green function

The expansion of G is obtained by exponentiating (6.1) as

$$G = e^{ik(x-y)} \left[1 + \frac{1}{2ik}b_1 + \frac{1}{(2ik)^2}b_2 + \cdots + \frac{1}{(2ik)^N}b_N + \Delta'_N \right], \tag{9.1}$$

where

$$b_n(x, y) = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\Sigma j_i = n} a_{j_1} a_{j_2} \cdots a_{j_m}, \tag{9.2}$$

$$\Delta'_N \equiv \exp\left[\sum_{m=1}^N \frac{a_m}{(2ik)^m} + \Delta_N \right] - \sum_{m=0}^N \frac{b_m}{(2ik)^m}. \tag{9.3}$$

(Here the prime does not denote a derivative.) It is obvious that $\Delta'_N = o(1/|k|^N)$ as $|k| \to \infty$ if $\Delta_N = o(1/|k|^N)$.

The time-dependent Green function for the Fokker–Planck equation is obtained from $G(x, y; k)$ by the inverse Fourier transformation as

$$G_F(x, y; t) = -\frac{1}{2\pi} e^{-[V(x)-V(y)]/2} \int_{-\infty}^{\infty} \frac{1}{2i\kappa} G(x, y; \kappa) \, e^{-i\omega t} \, d\omega, \tag{9.4}$$

where κ is defined by

$$\kappa^2 \equiv i\omega, \qquad \operatorname{Im}\kappa \geqslant 0. \tag{9.5}$$

(When necessary, the integral in (9.4) is to be understood as $\int_{-\infty+i\epsilon}^{\infty+i\epsilon}$ with positive infinitesimal ϵ). By substituting (9.1) into (9.4) and carrying out the integration term by term, we can derive an expansion of G_F in powers of t. The result is

$$G_F(x, y; t) = \frac{e^{-[V(x)-V(y)]/2}}{\sqrt{4\pi t}} \exp\left[-\frac{(x-y)^2}{4t}\right](1 + g_1 t + g_2 t^2 + g_3 t^3 + \cdots + g_N t^N + \tilde{\Delta}_N), \tag{9.6}$$

where

$$g_n(x, y) = \sum_{m=1}^{n} \frac{(-1)^n (2n - m - 1)!}{(m-1)!(n-m)!} \frac{b_m(x, y)}{(x-y)^{2n-m}}. \tag{9.7}$$

(See appendix B for the derivation. The expression for $\tilde{\Delta}_N$ is shown there). When $x \neq y$, the remainder term $\tilde{\Delta}_N$ satisfies

$$\lim_{t \to 0} \frac{\tilde{\Delta}_N(x, y; t)}{t^N} = 0 \tag{9.8}$$

if (6.4) is satisfied for $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$. (See appendix B). For (9.8) to hold, it is not necessary that (6.4) hold for $\operatorname{Im} k = 0$. So this short-time expansion is valid even when $f(\pm\infty)$ are infinite. (Here we are assuming that t is real).

Note that the right-hand side of (9.7) is not infinite at $x = y$ in spite of the appearance of the $1/(x-y)^{2n-m}$. For $x = y$ we have

$$g_n(x, x) = \frac{b_{2n}(x, x)}{2^n (2n-1)!!}, \tag{9.9}$$

as can be directly calculated. The right-hand side of (9.7) approaches (9.9) as $y \to x$.

10. Application to the Schrödinger equation

From (4.5), we may note that s_n for $n \geqslant 2$ can be expressed in terms of the Schrödinger potential V_S (equation (1.5)) and its derivatives

$$s_2 = -V_S, \qquad s_3 = -V_S', \qquad s_4 = V_S^2 - V_S'', \qquad s_5 = \left(2V_S^2 - V_S''\right)', \text{ etc.} \tag{10.1}$$

This can be confirmed as follows. From (A.3c) of appendix A and (2.12), we can show that S_r satisfies the differential equation

$$\frac{\partial}{\partial x} S_r(x, k) = 2ik S_r(x, k)[1 - S_r(x, k)] + f(x)[1 - 2S_r(x, k)]. \tag{10.2}$$

Substituting (4.2) into (10.2), we obtain

$$s_n' = s_{n+1} - \sum_{j=1}^{n} s_j s_{n+1-j} - 2f s_n + f\delta_{n0}. \tag{10.3}$$

Hence it follows that the s_n's satisfy the recursion relation

$$s_{n+1} = s_n' + \sum_{j=2}^{n-1} s_j s_{n+1-j} \qquad (n \geqslant 2), \tag{10.4}$$

where we have used $s_1 = -f$. We can obtain $s_3, s_4, s_5, \ldots$ from this recursion relation, starting with $s_2 = -V_S$. Therefore, any s_n $(n \geqslant 2)$ can indeed be expressed in terms of V_S and its derivatives. Substituting (10.1) into (6.2), we have

$$a_1 = \int_y^x V_S(z)\,dz, \qquad a_2 = -V_S(x) - V_S(y), \qquad a_3 = -\int_y^x \left[V_S^2(z) - V_S''(z)\right]dz,$$
$$a_4 = 2V_S^2(x) - V_S''(x) + 2V_S^2(y) - V_S''(y), \text{ etc.} \tag{10.5}$$

Substituting (10.5) into (6.1), and returning to definition (2.7), we can write

$$\log G_S(x, y; E) = -\log 2i - \log k + ik(x - y) + \frac{1}{2ik}\int_y^x V_S(z)\,dz$$
$$- \frac{1}{(2ik)^2}\left[V_S(x) + V_S(y)\right] - \frac{1}{(2ik)^3}\int_y^x \left[V_S^2(z) - V_S''(z)\right]dz + \cdots. \tag{10.6}$$

The recursion relation (10.4) is familiar in soliton theory. The quantities s_n obtained from it are, apart from the sign, identical to the conserved charge densities for the KdV equation [14, 20]. It is known that these quantities also appear in the asymptotic expansion for Jost solutions [16, 21]. Our results for the expansion of the Green function are valid even when Jost solutions do not exist.

So far, we have been assuming that the Schrödinger equation was derived from the Fokker–Planck equation (1.3). Let us now go in the opposite direction. To transform a given Schrödinger equation into a Fokker–Planck equation, it is, in general, necessary to shift the energy level. Let $\psi_0(x)$ be a solution of (1.4) with $E = E_0$. Then the Schrödinger equation (1.4) is equivalent to the Fokker–Planck equation

$$-\frac{d^2}{dx^2}\phi(x) + 2\frac{d}{dx}[f(x)\phi(x)] = p^2\phi(x), \tag{10.7}$$

where

$$\phi(x) = \psi_0(x)\psi(x), \qquad f(x) = \frac{d}{dx}\log\psi_0(x), \qquad p \equiv \sqrt{k^2 - E_0}, \qquad k^2 = E. \tag{10.8}$$

The relation between V_S and f is now

$$V_S(x) - E_0 = f^2(x) + f'(x). \tag{10.9}$$

So (1.5) is a special case of (10.9) with $E_0 = 0$. If we want $f(x)$ to be real and finite for any finite x, we need to choose E_0 such that $\psi_0(x) > 0$ for any finite x. We may take E_0 to be the ground state energy if V_S has a bound state.

Equation (10.6) was derived by using (1.5), which corresponds to $E_0 = 0$. However, since the Schrödinger equation (1.4) is invariant under the replacements $V_S \to V_S - E_0$ and $k \to p$, expansion (10.6) is valid even when $E_0 \neq 0$. We can check this by an explicit calculation. Suppose that $E_0 \neq 0$. Then it is obvious that a correct expression for $\log G_S$ is obtained by making the replacements $V_S \to V_S - E_0$ and $k \to p$ in (10.6) as

$$\log G_S(x, y; E) = -\log 2i - \log p + ip(x - y) + \frac{1}{2ip}\int_y^x [V_S(z) - E_0]\,dz$$
$$- \frac{1}{(2ip)^2}[V_S(x) + V_S(y) - 2E_0] + \cdots. \tag{10.10}$$

If we rearrange (10.10) into an expansion in powers of $1/k$ by substituting

$$\log p = \log k - \frac{E_0}{2k^2} - \frac{E_0^2}{4k^4} + \cdots, \qquad p = k - \frac{E_0}{2k} - \frac{E_0^2}{8k^3} + \cdots, \qquad \frac{1}{p} = \frac{1}{k} + \frac{E_0}{2k^3} + \cdots,$$

and so on, then it becomes (10.6). Thus, (10.6) holds for $E_0 \neq 0$ as well. Hence we know that (6.1), too, is valid for $E_0 \neq 0$. (See example 8 of the next section. Note that G is not $2ipG_S$ but $2ikG_S$.) Unlike the coefficients a_n, the remainder term Δ_N of (6.1) is not expressed solely in terms of V_S. But the conditions for the validity of (6.4) remain unchanged when $E_0 \neq 0$. We can understand this by writing (10.10) with the remainder term, and then turning it into an expansion in powers of $1/k$ as above.

Is is not difficult to interpret the results of sections 7 and 8 in the language of the Schrödinger equation. We see from the second equation of (10.8) that the conditions on f can be interpreted as the conditions on ψ_0. For example, $f(\infty)$ takes a nonzero finite value when the ground state wavefunction decays like e^{-cx} as $x \to \infty$. The differentiability conditions on f can be directly related to the differentiability of V_S. Namely, f is n times differentiable if V_S is $(n-1)$ times differentiable.

11. Examples

Let us consider some simple potentials for which the exact Green function can be obtained, and compare the exact expression with our results for the expansion

$$\log G(x, y; k) = ik(x - y) + \frac{a_1}{2ik} + \frac{a_2}{(2ik)^2} + \frac{a_3}{(2ik)^3} + \cdots. \tag{11.1}$$

The coefficients a_n are obtained from (6.2), (4.3) and (3.8). Explicit forms of the quantities s_n (equation (4.3)) are shown in (4.5) for $n \leqslant 5$. (Alternatively, we may use expressions (10.5) in terms of V_S.) We omit the derivation of the exact expressions.

Example 1. $V(z) = 2z,$ $\qquad f(z) = -1,$ $\qquad V_S(z) = 1.$

As the simplest example, let us first consider a linear potential. Substituting $f = -1$ into (4.5), and then into (6.2), we obtain the coefficients a_n as

$$a_1 = x - y, \qquad a_2 = -2, \qquad a_3 = -x + y, \qquad a_4 = 4, \text{ etc.} \tag{11.2}$$

The exact Green function for this potential is

$$G(x, y; k) = \frac{k}{\sqrt{k^2 - 1}} \exp[i\sqrt{k^2 - 1}(x - y)]. \tag{11.3}$$

It is easy to see that (11.1) with (11.2) is the correct expansion of the logarithm of (11.3) for $\mathrm{Im}\, k \geqslant 0$. Since the exact $\log G(k) - ik(x - y)$ does not have any singularities in $|k| > 1$, the right-hand side of (11.1) is a convergent infinite series for $|k| > 1$.

The short-time expansion (9.6) with g_n calculated from (9.7) and (9.2) reads

$$G_F(x, y; t) = e^{-(x-y)} \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{(x - y)^2}{4t}\right] \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots\right). \tag{11.4}$$

The series in parentheses on the right-hand side is $\sum_{n=0}^{\infty}(-t)^n/n! = e^{-t}$, so this expansion is convergent for any t.

Example 2. $V(z) = z^2,$ $\qquad f(z) = -z,$ $\qquad V_S(z) = z^2 - 1.$

Next, we consider a parabolic potential. This example satisfies conditions (i) and (ii) of section 7 for any N but not (iii). From (6.2) and (4.5), we obtain

$$a_1 = \frac{x^3 - y^3}{3} - (x - y), \qquad a_2 = 2 - (x^2 + y^2),$$

$$a_3 = -\frac{x^5 - y^5}{5} + \frac{2(x^3 - y^3)}{3} + x - y, \qquad a_4 = 2(x^4 + y^4) - 4(x^2 + y^2), \text{ etc.} \tag{11.5}$$

(*a*)

Im[log$G$−ik(x−y)]

Re[log$G$−ik(x−y)]

(*b*)

Im[log$G$−ik(x−y)]
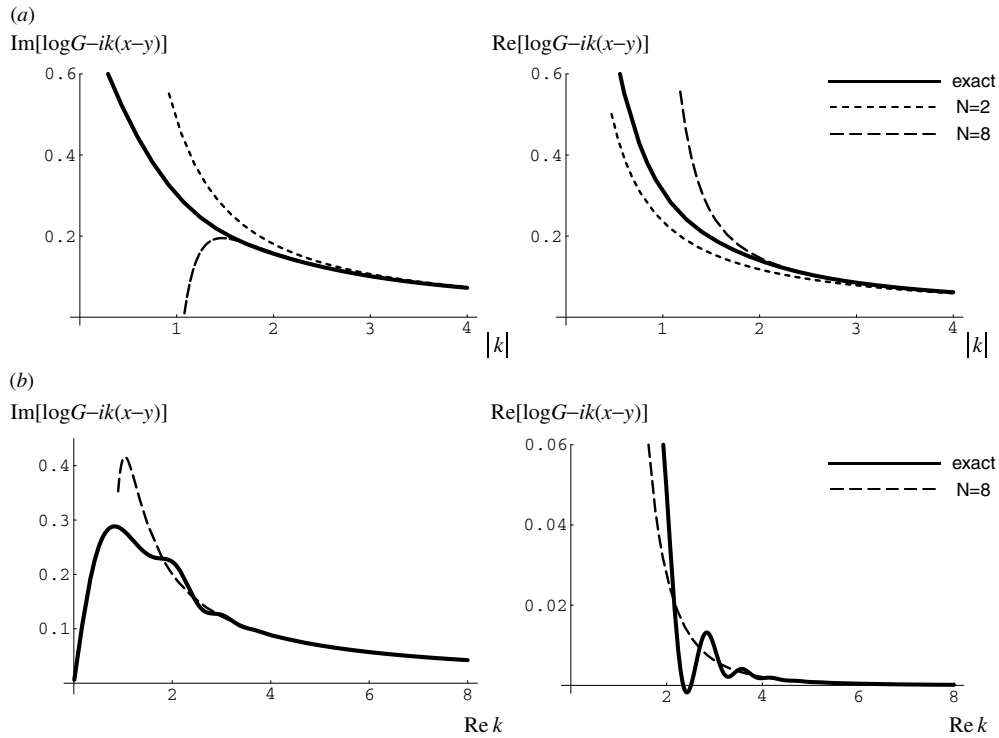
Re[log$G$−ik(x−y)]

Figure 1. The imaginary and real parts of $\log G(x, y; k) - \mathrm{i}k(x - y)$ for the potential $V(z) = z^2$ (example 2), with $x = 1$ and $y = 0$, (*a*) plotted as functions of $|k|$, with $\arg k$ fixed at $\pi/4$; (*b*) plotted as functions of $\mathrm{Re}\, k$, with $\mathrm{Im}\, k$ fixed at 0.75. Solid lines: the exact values. Dashed lines: expansion (11.1) to order $1/k^N$ ($N = 2$ and 8).

The exact Green function for this potential can be expressed as

$$G(x, y; k) = \frac{-\mathrm{i}k}{2\Gamma\left(-\frac{k^2}{4}\right)\Gamma\left(\frac{1}{2} - \frac{k^2}{4}\right)} \psi_+(x, k)\psi_-(y, k), \tag{11.6}$$

where

$$\psi_\pm(x, k) \equiv \mathrm{e}^{-x^2/2}\left[\Gamma\left(-\frac{k^2}{4}\right) F\left(-\frac{k^2}{4}, \frac{1}{2}; x^2\right) \mp 2x\Gamma\left(\frac{1}{2} - \frac{k^2}{4}\right) F\left(\frac{1}{2} - \frac{k^2}{4}, \frac{3}{2}; x^2\right)\right]. \tag{11.7}$$

Here Γ is the gamma function, and F is the confluent hypergeometric function defined by $F(\alpha, \gamma; z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)} \frac{1}{n!} z^n$. We can check that (11.1) with (11.5) is the correct asymptotic expansion of the logarithm of (11.6) when $\arg k$ is fixed in $0 < \arg k < \pi$ (figure 1(*a*)). Now (11.1) is divergent as an infinite series. This asymptotic expansion is also valid when $\mathrm{Im}\, k > 0$ is kept fixed as $|k| \to \infty$ (figure 1(*b*)). When $\mathrm{Im}\, k = 0$, however, (11.1) is not valid since $\log G(k)$ oscillates and does not vanish as $|k| \to \infty$.

The short-time expansion (9.6) now reads

$$G_\mathrm{F}(x, y; t) = \frac{1}{\sqrt{4\pi t}} \exp\left[\frac{-x^2 + y^2}{2} - \frac{(x - y)^2}{4t}\right] (1 + g_1 t + g_2 t^2 + g_3 t^3 + \cdots), \tag{11.8}$$

where the g_n's are obtained from (9.7), (9.2) and (11.5) as

$$g_1 = 1 - \tfrac{1}{3}(x^2 + xy + y^2), \qquad g_2 = \tfrac{1}{6} - \tfrac{1}{3}(x^2 + xy + y^2) + \tfrac{1}{18}(x^2 + xy + y^2)^2,$$

$$g_3 = -\tfrac{1}{6} + \tfrac{1}{15}xy + \tfrac{1}{30}(x^2 + xy + y^2) + \tfrac{1}{18}(x^2 + xy + y^2)^2 - \tfrac{1}{162}(x^2 + xy + y^2)^3. \tag{11.9}$$

The exact time-dependent Green function has the well-known form

$$G_F(x, y; t) = \left(\frac{1}{\pi[1 - \exp(-4t)]} \right)^{1/2} \exp\left(-\frac{[x - y\exp(-2t)]^2}{1 - \exp(-4t)} \right). \tag{11.10}$$

The expansion of (11.10) indeed has the form (11.8) with (11.9). This $G_F(t)$ has singularities in the complex t plane where $\exp(-4t) = 1$. The nearest singularities to the origin are at $t = \pm\pi i/2$. So, the infinite series (11.8) is convergent for $t < \pi/2$. This is a typical case where the short-time expansion is convergent for small t although the high-energy expansion, from which it was derived, is divergent.

Example 3. $V(z) = e^z, \qquad f(z) = -e^z/2, \qquad V_S(z) = (e^{2z}/4) - (e^z/2).$

This exponential potential satisfies conditions (i) of section 7 for any N but not (ii) or (iii). For this potential we have

$$a_1 = -\tfrac{1}{2}(e^x - e^y) + \tfrac{1}{8}(e^{2x} - e^{2y}), \qquad a_2 = \tfrac{1}{2}(e^x + e^y) - \tfrac{1}{4}(e^{2x} + e^{2y}),$$

$$a_3 = -\tfrac{1}{2}(e^x - e^y) + \tfrac{3}{8}(e^{2x} - e^{2y}) + \tfrac{1}{12}(e^{3x} - e^{3y}) - \tfrac{1}{64}(e^{4x} - e^{4y}), \text{ etc.} \tag{11.11}$$

The exact Green function has the form

$$G(x, y; k) = -\frac{i\pi}{2\cos(i\pi k)}[\chi_+(x, k) + e^{\pi k}\chi_-(x, k)]\chi_-(y, k), \tag{11.12}$$

where $\chi_\pm$ are defined in terms of the Bessel functions as

$$\chi_\pm(x, k) \equiv \frac{ie^{x/2}}{\sqrt{2}}\left[J_{\nu_\pm}(-ie^x/2) + iJ_{-\nu_\mp}(-ie^x/2) \right], \qquad \nu_\pm \equiv \frac{1}{2} \pm ik. \tag{11.13}$$

Equation (11.1) with (11.11) gives the correct asymptotic expansion when $\arg k$ is fixed in $0 < \arg k < \pi$ (figure 2). However, this asymptotic expansion is not correct when $|k| \to \infty$ with fixed $\operatorname{Im} k$, irrespective of whether $\operatorname{Im} k > 0$ or $\operatorname{Im} k = 0$. The short-time expansion (9.6) with g_n calculated from (11.11) is shown in figure 3.

Example 4. $V(z) = 2\log\cosh z, \qquad f(z) = -\tanh z, \qquad V_S(z) = 1 - 2\operatorname{sech}^2 z.$

This $V(x)$ tends to $+\infty$ linearly as $x \to \pm\infty$. Both $f(+\infty)$ and $f(-\infty)$ are finite, and conditions (iii) of section 7 are satisfied for any N. In this case, we have

$$a_1 = x - y - 2(\tanh x - \tanh y), \qquad a_2 = (\operatorname{sech} x)^2 + (\operatorname{sech} y)^2 - (\tanh x)^2 - (\tanh y)^2,$$

$$a_3 = -x + y + \tfrac{4}{3}(\tanh x - \tanh y) + \tfrac{8}{3}[(\operatorname{sech} x)^2\tanh x - (\operatorname{sech} y)^2\tanh y], \text{ etc.} \tag{11.14}$$

The exact G for this potential is

$$G(x, y; k) = \frac{k}{2\pi\sqrt{k^2 - 1}}\frac{\Gamma(\alpha)\Gamma(\tfrac{1}{2} - \alpha)\Gamma(\beta)\Gamma(\tfrac{1}{2} - \beta)\cos\alpha\pi\sin\beta\pi}{\Gamma(\alpha - \beta)\Gamma(\beta - \alpha)\sin[(\beta - \alpha)\pi]}\frac{\eta_+(x)\eta_-(y)}{\cosh x\cosh y}, \tag{11.15}$$

where

$$\eta_\pm(x, k) \equiv F\left(\alpha, \beta, \frac{1}{2}; -\sinh^2 x \right)$$

$$\mp 2\frac{\Gamma(\tfrac{1}{2} + \alpha)\Gamma(1 - \beta)}{\Gamma(\alpha)\Gamma(\tfrac{1}{2} - \beta)}\sinh x\, F\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2}, \frac{3}{2}; -\sinh^2 x \right), \tag{11.16}$$

(a)

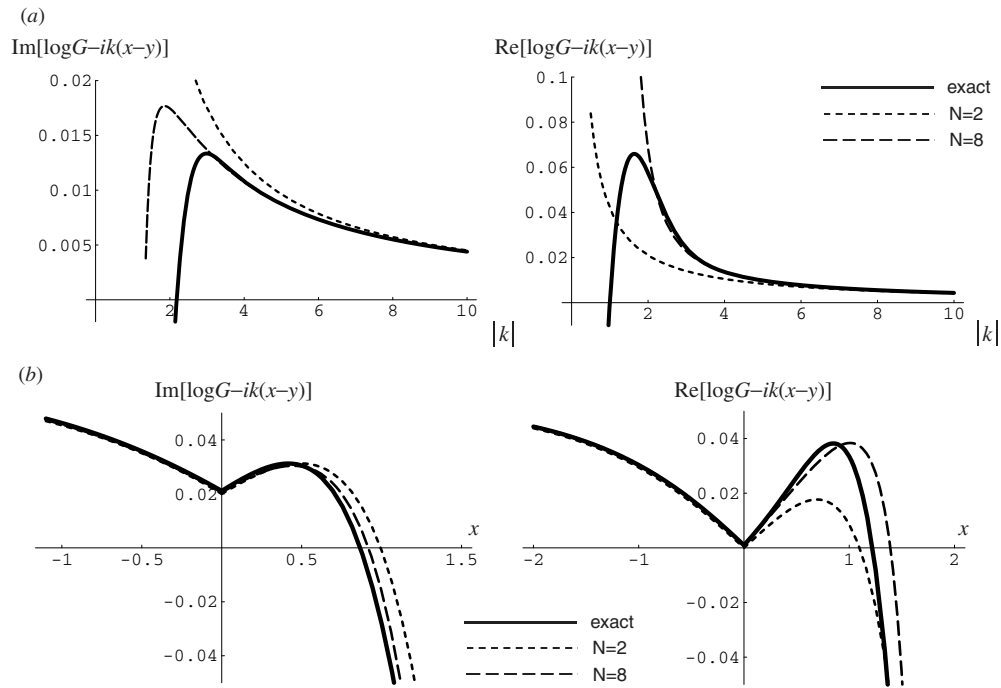Im[log G−ik(x−y)]

Re[log G−ik(x−y)]

Figure 2. The imaginary and real parts of $\log G(x, y; k) - ik(x - y)$ for the potential $V(z) = e^z$ (example 3), (a) plotted as functions of $|k|$, with $\arg k$ fixed at $\pi/4$; $x = 0.8$, $y = 0$; (b) plotted as functions of x, with $k = 2.5 \exp(i\pi/4)$, $y = 0$. Solid lines: the exact values. Dashed lines: expansion (11.1) to order $1/k^N$ ($N = 2$ and 8).
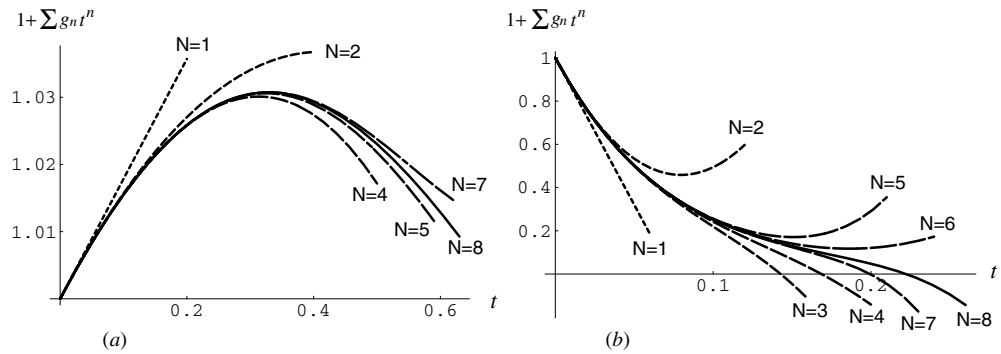
$1 + \sum g_n t^n$

$1 + \sum g_n t^n$

Figure 3. The series $1 + \sum_{n=1}^{N} g_n t^n$ for the potential $V(z) = e^z$ (example 3), plotted as a function of t with various N. (a) $x = 0.7$, $y = 0$; (b) $x = 3$, $y = 0$.

with $\alpha \equiv \frac{1}{2}(-1 - i\sqrt{k^2 - 1})$ and $\beta \equiv \frac{1}{2}(-1 + i\sqrt{k^2 - 1})$. Here F is the hypergeometric function defined by $F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{1}{n!} z^n$. Since $f(\pm\infty)$ are finite, expansion (11.1) with (11.14) is valid even when $\text{Im} \, k = 0$ (figure 4). As in example 1, the infinite series (11.1) is convergent for $|k| > 1$.
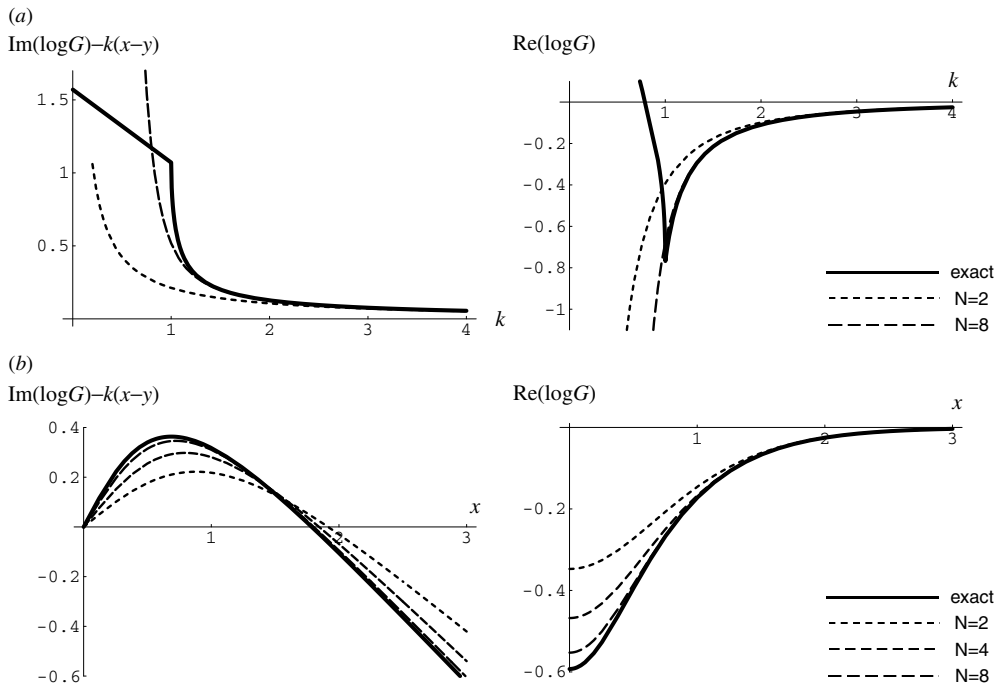
(*a*)

(*b*)

Figure 4. The imaginary and real parts of $\log G(x, y; k) - ik(x - y)$ for the potential $V(z) = 2\log\cosh z$ (example 4), (*a*) plotted as functions of real k, with $x = 0.5$, $y = 0$; (*b*) plotted as functions of x, with $k = 1.2$, $y = 0$. Solid lines: the exact values. Dashed lines: expansion (11.1) to order $1/k^N$. (Since k is real, $N = 2, 4, 8$ are the same as $N = 1, 3, 7$, respectively, for the imaginary part.)

Example 5. $V(z) = 2|z|$, $\qquad f(z) = 1 - 2\theta(z)$, $\qquad V_S(z) = 1 - 2\delta(z)$.

This is an example where $f(z)$ has a jump at $z = 0$. This belongs to the case of (7.1) with $M = 0$ and $C = -2$. For $y < 0 < x$, the exact Green function is

$$G(x, y; k) = \frac{i + \sqrt{k^2 - 1}}{k} \exp\left[i\sqrt{k^2 - 1}(x - y)\right], \qquad (11.17)$$

and hence we have, as $|k| \to \infty$ $(0 \leqslant \arg k \leqslant \pi)$,

$$\log G(x, y; k) = ik(x - y) + \frac{1}{2ik}(x - y - 2) - \frac{1}{(2ik)^3}\left(x - y - \frac{4}{3}\right) + O(1/|k|^5). \quad (11.18)$$

On the other hand, a_1 and a_2 are obtained from (10.5) as

$$a_1 = \int_y^x [1 - 2\delta(z)]\,dz = x - y - 2, \qquad a_2 = -1 + 2\delta(x) - 1 + 2\delta(y) = -2. \quad (11.19)$$

Comparing (11.19) with (11.18), we can see that a_1 is the correct coefficient of the expansion but a_2 is not. As shown in section 8, the coefficient a_2 needs to be corrected by $C^2/2$. Since $a_2 + C^2/2 = 0$, we can see that (8.10) indeed agrees with (11.18).

Example 6.

$$V(z) = \begin{cases} e^z \\ 1 + z, \end{cases} \qquad f(z) = \begin{cases} -e^z/2 \\ -1/2, \end{cases} \qquad V_S(z) = \begin{cases} (e^{2z}/4) - (e^z/2) & (z < 0) \\ 1/4 & (z > 0). \end{cases}$$
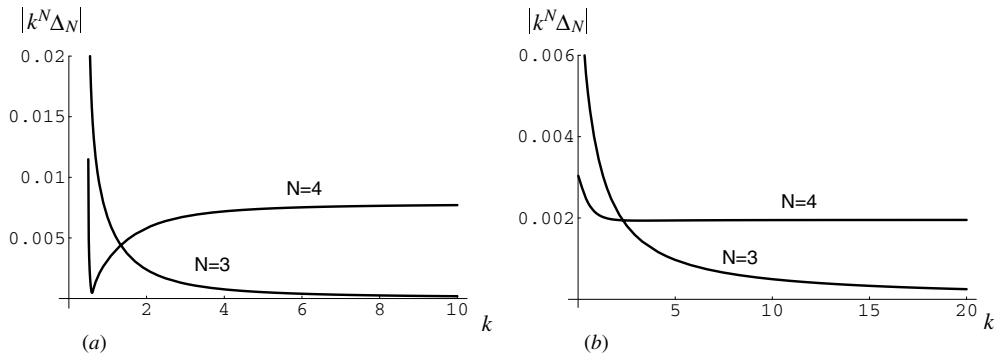
Figure 5. The graphs of $|k^N \Delta_N(x, y; k)|$ for examples 6 and 7, as functions of real k. ($N = 3$ and 4.) (*a*) Example 6: $x = 0.5$, $y = -0.5$. (*b*) Example 7: $x = 0.5$, $y = -0.5$.

This example belongs to the case of (7.1) with $M = 1$ and $C = -1/2$. Now $f(z)$ is continuous and $f'(z)$ has a jump at $z = 0$. For $y < 0 < x$, the exact G has the form

$$G(x, y; k) = \frac{-2\sqrt{2}\mathrm{i}k\,\mathrm{e}^{\mathrm{i}Kx}\chi_-(y, k)}{[K - \mathrm{i}\nu_-]J_{\nu_-}(-\mathrm{i}/2) + (\mathrm{i}K + \nu_+)J_{-\nu_+}(-\mathrm{i}/2)} \tag{11.20}$$

with $K \equiv \sqrt{k^2 - (1/4)}$, where χ_- and $\nu_\pm$ are defied by (11.13). As shown in figure 5(*a*), we can check that (6.4) holds for $N \leqslant 3$ but not for $N \geqslant 4$. It can be seen that the limit of $|k^4 \Delta_4|$ as $|k| \to \infty$ is $2^{-5}C^2 \simeq 0.0078$ as predicted by (8.9). The coefficients of the expansion for $y < 0 < x$ are obtained from (6.2) as

$$a_1 = \tfrac{1}{8}(-\mathrm{e}^{2y} + 4\mathrm{e}^y + 2x - 3), \qquad a_2 = -\tfrac{1}{4}(\mathrm{e}^{2y} - 2\mathrm{e}^y + 1),$$
$$a_3 = \tfrac{1}{192}(3\mathrm{e}^{4y} - 16\mathrm{e}^{3y} - 72\mathrm{e}^{2y} + 96\mathrm{e}^y - 12x - 11), \tag{11.21}$$
$$a_4 = \tfrac{1}{8}(\mathrm{e}^{4y} - 4\mathrm{e}^{3y} - 4\mathrm{e}^{2y} + 4\mathrm{e}^y + 1), \text{ etc.}$$

Since both $f(-\infty)$ and $f(+\infty)$ are finite, expansion (11.1) with (11.21) is correct to order $1/k^3$ for $\mathrm{Im}\,k \geqslant 0$. The correct expansion to order $1/k^4$ is obtained by adding $-C^2/2 = -1/8$ to a_4 as in (8.10). (See figure 6(*a*).)

For $0 < y < x$, the exact Green function is

$$G(x, y; k) = \frac{k}{K}\left(\mathrm{e}^{-\mathrm{i}Ky} + \frac{A_-}{A_+}\mathrm{e}^{\mathrm{i}Ky}\right)\mathrm{e}^{\mathrm{i}Kx}, \tag{11.22}$$

where

$$A_\pm \equiv \left(\mathrm{i} \pm \frac{\nu_-}{K}\right)J_{\nu_-}(-\mathrm{i}/2) - \left(1 \mp \mathrm{i}\frac{\nu_+}{K}\right)J_{-\nu_+}(-\mathrm{i}/2). \tag{11.23}$$

The a_n obtained from (6.2) for $0 < y < x$ are

$$a_1 = \tfrac{1}{4}(x - y), \qquad a_2 = -\tfrac{1}{2}, \qquad a_3 = -\tfrac{1}{16}(x - y), \text{ etc.} \tag{11.24}$$

We can show that the quantity A_-/A_+ is $O(1/|k|^2)$ as $|k| \to \infty$ ($0 \leqslant \arg k \leqslant \pi$). Therefore, from (11.22) we can see that (11.1) with (11.24) is correct only up to order $1/k$ when $\mathrm{Im}\,k = 0$ (figure 6(*b*)). For $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$, this asymptotic expansion is correct to any order since $\mathrm{e}^{\mathrm{i}Ky}$ in (11.22) vanishes faster than any power of $1/k$.
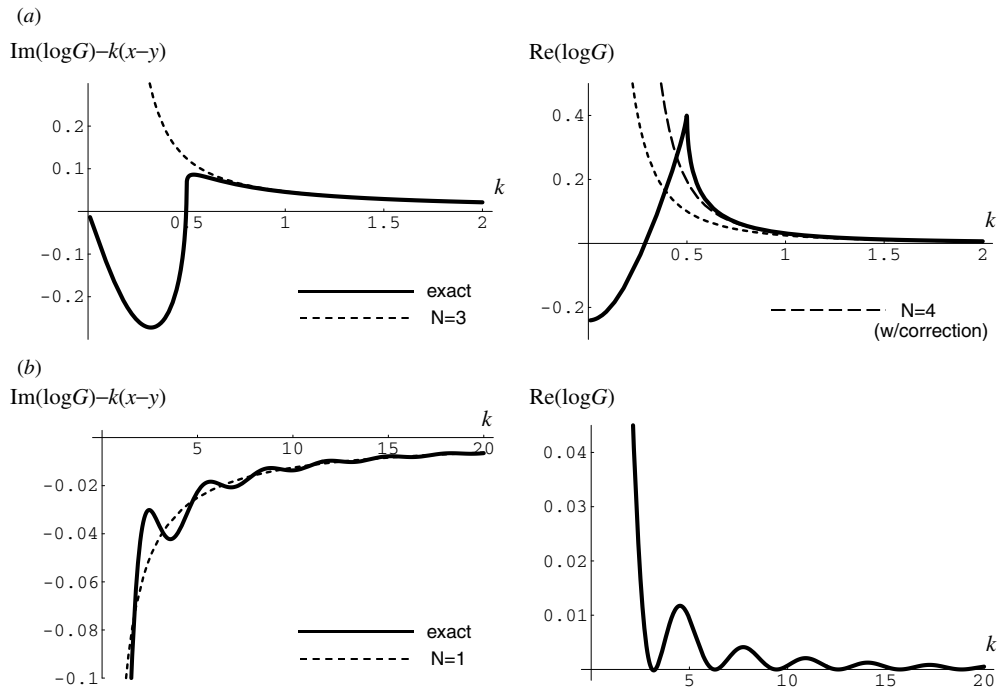
(a)

(b)

Figure 6. The imaginary and real parts of $\log G(x, y; k) - ik(x - y)$ for the potential of example 6, plotted as functions of real k; (a) $x = 0.5$, $y = -1$; (b) $x = 2$, $y = 1$. In (a), the dashed curve labelled '$N = 4$' shows the expansion to order $1/k^4$ with the correction term $-C^2/2$ added to a_4 as in equation (8.10). In (b), the envelope of the oscillation of $\mathrm{Re}(\log G)$ falls off like $1/k^2$.

Example 7.

$$V(z) = (\mathrm{sgn}\, z)[(1 + |z|)^{1/2} - 1], \qquad f(z) = -\tfrac{1}{4}(1 + |z|)^{-1/2},$$
$$V_S(z) = \tfrac{1}{16}(1 + |z|)^{-1} + \tfrac{1}{8}(\mathrm{sgn}\, z)(1 + |z|)^{-3/2}.$$

This is another case where $f'(z)$ has a finite jump at $z = 0$. The exact Green function for $y < 0 < x$ can be expressed in terms of confluent hypergeometric functions as

$$G(x, y; k) = \frac{2ik\zeta_+(x, k)\zeta_-(y, k)}{\zeta_+'(0, k)\zeta_-(0, k) - \zeta_+(0, k)\zeta_-'(0, k)}, \tag{11.25}$$

where

$$\zeta_\pm(x, k) \equiv \mathrm{e}^{ik(1\pm x)}\left\{ q^{1/2}\Gamma(q)\left[F\left(q, \frac{1}{2}; -2ik(1 \pm x)\right)\right.\right.$$

$$\left.\mp \frac{\sqrt{1 \pm x}}{2} F\left(q + 1, \frac{3}{2}; -2ik(1 \pm x)\right)\right]$$

$$\pm \Gamma\left(q + \frac{1}{2}\right)\left[F\left(q + \frac{1}{2}, \frac{1}{2}; -2ik(1 \pm x)\right)\right.$$

$$\left.\left.\mp \frac{\sqrt{1 \pm x}}{2} F\left(q + \frac{1}{2}, \frac{3}{2}; -2ik(1 \pm x)\right)\right]\right\} \tag{11.26}$$
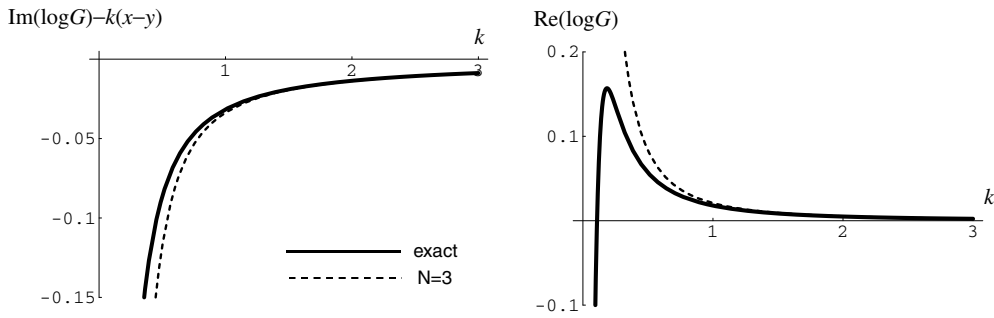
Figure 7. The imaginary and real parts of $\log G(x, y; k) - ik(x - y)$ for the potential of example 7, plotted as functions of real k; $x = 0.5$, $y = -0.5$.

with $q \equiv i/(32k)$, and $\zeta'_\pm(x, k) \equiv (\partial/\partial x)\zeta_\pm(x, k)$. For $y < 0 < x$, the coefficients of the expansion calculated by our method are

$$a_1 = \tfrac{1}{16}[\log(1 + x) + \log(1 - y)] - \tfrac{1}{4}[(1 + x)^{-1/2} - (1 - y)^{-1/2}],$$
$$a_2 = -\tfrac{1}{16}[(1 + x)^{-1} + (1 - y)^{-1}] - \tfrac{1}{8}[(1 + x)^{-3/2} - (1 - y)^{-3/2}], \text{ etc.}$$
(11.27)

This, too, is a case of (7.1) with $M = 1$, and so (6.4) holds for $N \leqslant 3$ but not for $N \geqslant 4$. Expansion (11.1) is now correct to order $1/k^3$ for $\text{Im}\, k \geqslant 0$ (figure 7). We can see from figure 5(b) that $|k^4 \Delta_4|$ indeed approaches the predicted value $2^{-5}C^2 \simeq 0.0020$ as $|k| \to \infty$. (In this case, $C = 1/4$.)

Example 8. $V_S(z) = |z|$.

Here we consider the case where V_S is given, and where $E_0 \neq 0$ (see section 10). For $y < 0 < x$, the coefficients a_n are obtained from (10.5) (or (10.4) and (6.2)) as

$$a_1 = \tfrac{1}{2}(x^2 + y^2), \qquad a_2 = -x + y, \qquad a_3 = 2 - \tfrac{1}{3}(x^3 - y^3), \qquad a_4 = 2(x^2 + y^2),$$
$$a_5 = \tfrac{1}{2}(x^4 + y^4) - 5(x - y), \qquad a_6 = -\tfrac{16}{3}(x^3 - y^3) + 10, \text{ etc.}$$
(11.28)

The exact Green function for $y < 0 < x$ is

$$G(x, y; k) = \frac{ik\,\text{Ai}(x - k^2)\text{Ai}(-y - k^2)}{\text{Ai}(-k^2)\text{Ai}'(-k^2)},$$
(11.29)

where $\text{Ai}(z)$ is the Airy function and $\text{Ai}'(z)$ is its derivative. From the second equation of (10.8) we have $f(z) = (\mathrm{d}/\mathrm{d}z) \log \text{Ai}(|z| - E_0)$, where E_0 is the smallest number satisfying $\text{Ai}'(-E_0) = 0$. (Numerically, $E_0 \simeq 1.019$.) It is easy to see that $f''(0+) - f''(0-) = 2$, so that this is a case of (7.1) with $M = 2$, $C = 2$. Expansion (11.1) with (11.28) is correct to order $1/k^5$ for $\text{Im}\, k \geqslant \epsilon$. (Conditions (ii) of section 7 are satisfied since $f(z)$ behaves like $|z|^{1/2}$ as $|z| \to \infty$. This expansion is not valid for $\text{Im}\, k = 0$.) We can also check that the correct coefficient of order $1/k^6$ is not a_6 but $a_6 + C^2/2 = a_6 + 2$.

12. Summary and remarks

In this paper, we studied the high-energy asymptotic behaviour of the Green function. The expansion of $\log G$ in powers of $1/k$ (with G defined by (2.7) and (2.5)) is given by (6.1).

The coefficients of the expansion (equations (6.2) and (4.3)) are expressed in terms of the coefficients $\tilde{c}_n$ for the expansion of the generalized reflection coefficient, which are calculated by using formula (3.8) with (3.2). The remainder term Δ_N is also expressed in terms of $\tilde{c}_n$ (see (6.3), (5.3), (4.12), (4.9), (4.6) and (3.9)). Sufficient conditions for the validity of the expansion to order N (equation (6.4)) are given by (i), (ii), (iii) of section 7. These are not necessary conditions. Equation (6.4) holds under broader conditions as shown in section 8.

We assumed that the potential $V(x)$ is monotone for sufficiently large $|x|$, but this is not an essential restriction for our formalism. The formulae for the coefficients of the expansion and the remainder term are valid for any potential, as long as these quantities make sense. The particular shape of the potential is relevant only to the conditions for the validity of (6.4). The above-mentioned assumption on the potential is used only in deriving the conditions for (3.12) quoted at the end of section 3. Even for other kinds of potentials, we can use the same method to derive the criterion for (6.4). For example, in this paper we excluded the cases where $V(x)$ oscillates indefinitely as $|x| \to \infty$, but our method can be applied to these cases as well, if only we study the validity of (3.12) (and its analogue for R_l) for such potentials in a similar way as in [19].

Appendix A. Properties of scattering coefficients for finite intervals

Here we summarize some properties of the transmission and reflection coefficients for finite intervals. (For details, see [22] and references therein.) First, it is obvious that

$$\tau(x, x; k) = 1, \qquad R_r(x, x; k) = 0, \qquad R_l(x, x; k) = 0. \tag{A.1}$$

Let us assume that f is piecewise smooth. We have, for finite x and y,

$$\tau(x, y; k) = e^{ik(x-y)}[1 + O(1/|k|)], \tag{A.2a}$$

$$R_r(x, y; k) = O(1/|k|), \qquad R_l(x, y; k) = O(1/|k|), \tag{A.2b}$$

as $|k| \to \infty$ ($0 \leqslant \arg k \leqslant \pi$). The first equation of (A.2b) is a special case of (3.3) with $N = 1$. For finite x and y, equations (A.2) can be derived more directly from integral representations of the scattering coefficients. (See equations (1.14) of [22]. Alternatively, we can use equations (3.8) of [22] and the asymptotic forms of the functions $\alpha^\pm$ and $\beta^\pm$ to derive (A.2).)

From equations (3.5) and (3.8) of [22], we can derive the differential equations

$$\frac{\partial}{\partial x} \tau(x, y; k) = ik\tau(x, y; k) - f(x)R_r(x, y; k)\tau(x, y; k), \tag{A.3a}$$

$$\frac{\partial}{\partial y} \tau(x, y; k) = -ik\tau(x, y; k) - f(y)R_l(x, y; k)\tau(x, y; k), \tag{A.3b}$$

$$\frac{\partial}{\partial x} R_r(x, y; k) = 2ikR_r(x, y; k) + f(x)\left[1 - R_r^2(x, y; k)\right], \tag{A.3c}$$

$$\frac{\partial}{\partial y} R_r(x, y; k) = -f(y)\tau^2(x, y; k), \tag{A.3d}$$

$$\frac{\partial}{\partial x} R_l(x, y; k) = -f(x)\tau^2(x, y; k), \tag{A.3e}$$

$$\frac{\partial}{\partial y} R_l(x, y; k) = -2ikR_l(x, y; k) + f(y)\left[1 - R_l^2(x, y; k)\right]. \tag{A.3f}$$

From (A.2) and (A.3) it follows that, as $|k| \to \infty$ $(0 \leqslant \arg k \leqslant \pi)$,

$$\frac{\partial}{\partial z}\left[\frac{\tau(x,z)}{1+R_r(x,z)}\right]^2 = -2ik\left[\frac{\tau(x,z)}{1+R_r(x,z)}\right]^2 + 2f(z)\,e^{4ik(x-z)} + o(1), \tag{A.4a}$$

$$\frac{\partial}{\partial z}\left[\frac{\tau(z,x)}{1+R_l(z,x)}\right]^2 = 2ik\left[\frac{\tau(z,x)}{1+R_l(z,x)}\right]^2 + 2f(z)\,e^{4ik(z-x)} + o(1), \tag{A.4b}$$

$$\int_a^b \left[\frac{\tau(z,c)}{1+R_r(z,c)}\right]^2 \mathrm{d}z = \frac{1}{2ik}\left\{\left[\frac{\tau(b,c)}{1+R_r(b,c)}\right]^2 - \left[\frac{\tau(a,c)}{1+R_r(a,c)}\right]^2\right\} + o(1/|k|), \tag{A.5a}$$

$$\int_a^b \left[\frac{\tau(c,z)}{1+R_l(c,z)}\right]^2 \mathrm{d}z = \frac{1}{2ik}\left\{-\left[\frac{\tau(c,b)}{1+R_l(c,b)}\right]^2 + \left[\frac{\tau(c,a)}{1+R_l(c,a)}\right]^2\right\} + o(1/|k|). \tag{A.5b}$$

Appendix B. Derivation of the short-time expansion

Let $X \equiv x - y$. When (9.1) is substituted into (9.4), the integral of the mth-order term is proportional to

$$\int_{-\infty}^{\infty} \frac{e^{i\kappa X}e^{-i\omega t}}{\kappa^{m+1}}\,\mathrm{d}\omega = -2i\frac{e^{-X^2/(4t)}}{\sqrt{t}}\int_{-\infty}^{\infty}\left(\frac{2t}{iX}\right)^m\left(1+\frac{2\sqrt{t}}{iX}p\right)^{-m} e^{-p^2}\mathrm{d}p, \tag{B.1}$$

where we have changed the variable of integration from ω to $p \equiv (\kappa/\sqrt{t}) - [iX/(t\sqrt{t})]$, and deformed the contour of integration. The right-hand side of (B.1) can be expanded in powers of t by using the formula of the Taylor expansion

$$(1+\alpha)^{-m} = \sum_{j=0}^{M}\frac{(m+j-1)!}{j!(m-1)!}(-1)^j\alpha^j + \frac{(M+m)!}{(M+1)!(m-1)!}\frac{(-1)^{M+1}\alpha^{M+1}}{(1+\theta\alpha)^{M+m+1}}, \tag{B.2}$$

where M is an arbitrary positive integer, and $0 < \theta < 1$. Applying (B.2) to (B.1), and carrying out the integration of each term, we have

$$\int_{-\infty}^{\infty}\frac{e^{i\kappa X}\,e^{-i\omega t}}{\kappa^{m+1}}\,\mathrm{d}\omega = -2i\frac{e^{-X^2/(4t)}}{\sqrt{t}}\left\{\sqrt{\pi}\sum_{j=0}^{M}\frac{2^{m+j}(2j+m-1)!}{(m-1)!(2j)!}\frac{t^{j+m}}{(iX)^{2j+m}}\right.$$

$$\left.+\frac{2^{m+2M+2}(2M+m+1)!}{(m-1)!(2M+2)!}\frac{t^{M+m+1}}{(iX)^{2M+m+2}}\int_{-\infty}^{\infty}\frac{p^{2M+2}\,e^{-p^2}\mathrm{d}p}{\left[1+\frac{2\sqrt{t}}{iX}\theta p\right]^{2M+m+2}}\right\}. \tag{B.3}$$

From (9.4), (9.1) and (B.3), we obtain expansion (9.6) with (9.7). The remainder term $\tilde{\Delta}_N$ can be written as

$$\tilde{\Delta}_N = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\Delta_N'\left(x,y;\frac{p}{\sqrt{t}}+\frac{iX}{2t}\right)e^{-p^2}\mathrm{d}p$$

$$+\frac{t^{N+1}}{\sqrt{\pi}}\sum_{m=1}^{N}\frac{b_m}{X^{2N-m+2}}\frac{(-1)^{N+1}2^{2N-2m+2}(2N-m+1)!}{(m-1)!(2N-2m+2)!}\int_{-\infty}^{\infty}\frac{p^{2N-2m+2}\,e^{-p^2}\mathrm{d}p}{\left[1+\frac{2\sqrt{t}}{iX}\theta_m p\right]^{2N-m+2}}, \tag{B.4}$$

where $0 < \theta_m < 1$. The second term of (B.4) is obviously $O(t^{N+1})$ as $t \to 0$. Assuming that $X \neq 0$, the first term is $o(t^N)$ as $t \to 0$ if $\Delta_N'(x,y;k) = o(1/|k|^N)$ as $|k| \to \infty$ in the region $\epsilon \leqslant \arg k \leqslant \pi - \epsilon$.

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